Spectral-Domain Calculation of Microstrip Characteristic Impedance

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***Abstract*—This paper presents a hybrid-mode solution for the characteristic impedance of microstrip on lossless dielectric substrate. A solution to the hybrid-mode equations is obtained by applying the method of moments in the Fourier transform domain. Numerical results are presented showing the frequency dependence of both wavelength and characteristic impedance for single and coupled strips. These results are compared with those of other investigators in the low-frequency range.**

I. INTRODUCTION

THE spectral-domain technique is a powerful, accurate, numerically efficient approach for analysis of planar transmission line structures. This technique was first suggested by Itoh and Mittra [1] and has been applied to calculate the dispersion characteristic of a single slot [1], the dispersion characteristic of a single microstrip [2], [3], and the resonant frequency of rectangular microstrip resonators [4] from which a calculation of microstrip open circuit end effect may also be obtained. The dispersion characteristic and characteristic impedances of coupled slots and coplanar strips have also been obtained using this approach [5].

Microstrip is a structure which has been studied by many investigators. There are numerous quasi-static analyses and a lesser number of frequency-dependent analyses which have been carried out. Among these frequency-dependent analyses is one by Krage and Haddad [6] which appears to be the only study to include an investigation of the frequency dependence of microstrip characteristic impedance. Results are presented for only a relatively low near-quasi-static frequency range ($\lambda > 0.1\lambda_d$), however.

The purpose of this paper is to present the results of a study of the frequency dependence of the characteristic impedance of microstrip using the spectral-domain approach. The method whereby the spectral-domain approach may be extended to calculate characteristic impedance will first be described. Numerical results showing the variation of characteristic impedance over a wide frequency range will then be presented, and it will be shown that these results converge to those of other investigators in the low-frequency range.

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II. DISPERSION CHARACTERISTICS OF MICROSTRIP ON A LOSSLESS DIELECTRIC SUBSTRATE

To calculate the characteristic impedance of microstrip by the spectral-domain approach, it is first necessary to calculate the dispersion characteristic. The following discussion is included to provide an introduction to the method of analysis and to further reference the results of this study to those of other authors.

The spectral-domain dispersion analysis of microstrip is discussed in [2] and will be outlined only briefly here. With reference to Fig. 1, the microstrip field is expressed as a linear combination of TE and TM modes characterized by

$$E_{zi}(x,y,z) = k_{ci}^2\phi_i^e(x,y)e^{\gamma z} \tag{1a}$$

$$H_{zi}(x,y,z) = k_{ci}^2\phi_i^h(x,y)e^{\gamma z} \tag{1b}$$

where $k_{ci}^2 = \gamma^2 + k_i^2$, i denotes the appropriate region, and the ϕ_i are unknown scalar potential functions. Applying boundary conditions at $y = 0$ and $y = D$ leads to a set of boundary equations which still contain the $\phi_i(x,y)$. Although the ϕ_i are unknown, their Fourier transforms, $\Phi_i(\alpha,y)$, with respect to x can be found, and thus the boundary equations are transformed and the general solutions for the $\Phi_i(\alpha,y)$ are substituted. Extensive algebraic manipulation of the resulting equations leads to the coupled set

$$G_1(\alpha,\beta)\mathcal{J}_x(\alpha) + G_2(\alpha,\beta)\mathcal{J}_z(\alpha) = \mathcal{E}_x(\alpha) \tag{2a}$$

$$G_3(\alpha,\beta)\mathcal{J}_x(\alpha) + G_4(\alpha,\beta)\mathcal{J}_z(\alpha) = \mathcal{E}_z(\alpha) \tag{2b}$$

where α is the transform variable and $\mathcal{E}_i(\alpha)$ and $\mathcal{J}_i(\alpha)$ are the transforms of the electric field and the surface current at $y = D$. We next define the inner product

$$\langle A(\alpha),B(\alpha)\rangle = \int_{-\infty}^{+\infty} A(\alpha)B^*(\alpha)\,d\alpha \tag{3}$$

and take the inner product of (2a) and (2b) with weighting functions $W_i(\alpha)$. If we choose

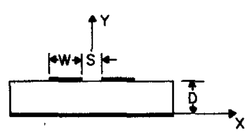

Fig. 1. Microstrip on a dielectric substrate.

$$W_1(\alpha) = \mathcal{J}_x(\alpha) \quad (4a)$$

$$W_2(\alpha) = \mathcal{J}_z(\alpha) \quad (4b)$$

we obtain

$$\langle G_1\mathcal{J}_x, \mathcal{J}_x\rangle + \langle G_2\mathcal{J}_z, \mathcal{J}_x\rangle = 0 \quad (5a)$$

$$\langle G_3\mathcal{J}_x, \mathcal{J}_z\rangle + \langle G_4\mathcal{J}_z, \mathcal{J}_z\rangle = 0. \quad (5b)$$

That the right-hand side of these equations is zero follows from Parseval's theorem since electric field and surface current at $y = D$ are orthogonal in the space domain.

Equations (5a) and (5b) are exact. The $G_i(\alpha,\beta)$ reflect substrate thickness, dielectric constant, and frequency while strip widths and current distributions determine the $\mathcal{J}$'s. A moment solution of (5) can be obtained by expanding $\mathcal{J}_x$ and $\mathcal{J}_z$ in a known set of basis functions and solving the resulting determinant. Various choices of bases have been considered by the authors and in [2]. Accurate results are obtained by neglecting transverse current ($\mathcal{J}_x(\alpha) \approx 0$) and assuming that longitudinal current is uniformly distributed. For simplicity and computational efficiency, the current has been assumed z directed and uniformly distributed in this study.

Fig. 2 shows the free space-to-microstrip wavelength ratio for a single strip and Fig. 3 shows the same ratio for the odd and even modes of coupled strips. Also shown are theoretical results published by Krage and Haddad [6], Wheeler [7], and Bryant and Weiss [8]. Where the present results overlap with those of [6], very good agreement is evident which tends to confirm the accuracy of both methods. The frequency-dependent analyses show an increasing free space-to-microstrip wavelength ratio with increasing frequency due to the relatively higher proportion of power in the dielectric. The inaccuracy of the quasi-static results at high frequencies is evident.

It is always desirable to compare theory with experiment, and this comparison appears in Figs. 4 and 5. In Fig. 4 the theoretical effective dielectric constant of a single microstrip is compared with data published by Getsinger [9]. In Fig. 5 the theoretical wavelength ratios for the odd and even modes of coupled microstrips are compared with data published by Gould and Tolboys [10]. In all cases the agreement between theory and experiment is better than 2 percent although the data from [10] show a constant offset. We cannot offer any explanation for this discrepancy. Getsinger [11] has obtained a somewhat better fit to these same data by using his approximate dispersion relation, but uses an empirical factor to do so.

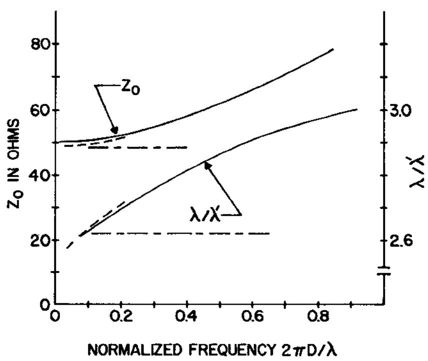

Fig. 2. Wavelength and characteristic impedance versus frequency for a single microstrip. $\epsilon_r = 10$, $W/D = 1$. Present method: ——; Krage and Haddad: ——; Wheeler: – —.

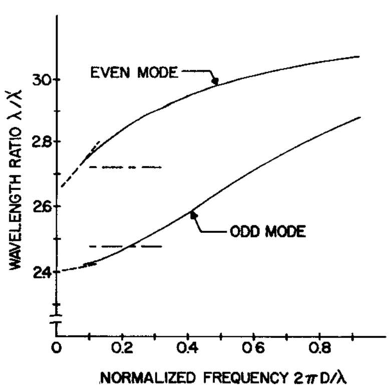

Fig. 3. Free space-to-microstrip wavelength ratio versus frequency for coupled microstrips. $\epsilon_r = 10$, $W/D = 1$, $S/D = 0.4$. Present method: ——; Krage and Haddad: ——; Bryant and Weiss: – —.

III. CHARACTERISTIC IMPEDANCE OF MICROSTRIP ON A LOSSLESS DIELECTRIC SUBSTRATE

The extension of the spectral-domain technique to calculate the characteristic impedance of microstrip proceeds as follows. We again neglect transverse current and define

$$Z_{0i} = \frac{2P_{\text{avg}}}{I_{0z}^2} \quad (6a)$$

for a single strip or

$$Z_{0i} = \frac{P_{\text{avg}}}{I_{0z}^2} \quad (6b)$$

for coupled strips with reflection symmetry where I_{0z} is the total z-directed strip current. The average power is cal-

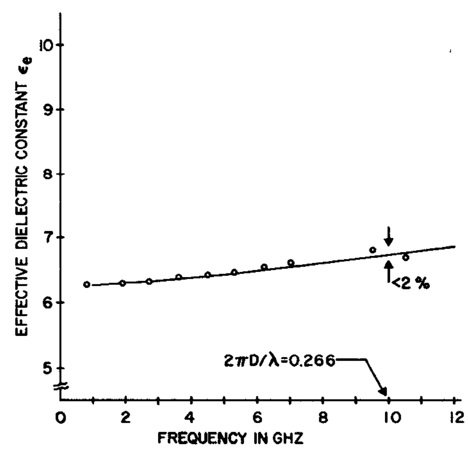

Fig. 4. Effective dielectric constant versus frequency for a single microstrip. $\epsilon_r = 10.185$, $W/D = 0.2$. $D = 0.050$ in. Present method: ——; Getsinger's data: ○.

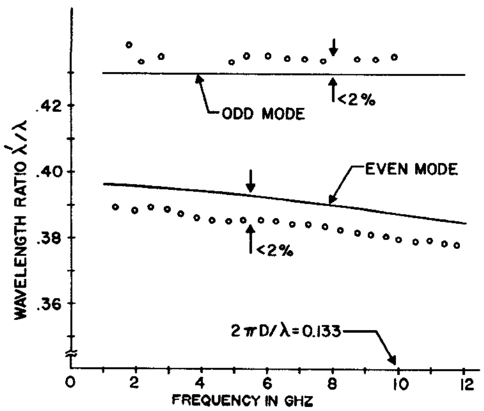

Fig. 5. Microstrip-to-free space wavelength ratio versus frequency for coupled microstrips. $\epsilon_r = 9.7$, $W/D = 0.3$, $D = 0.025$ in. Present method: ——; Gould and Tolboys' data: ○.

culated as

$$P_{\text{avg}} = \tfrac{1}{2} \operatorname{Re} \iint_s (E_x H_y^* - E_y H_x^*) \, dx \, dy \tag{7}$$

where the transverse fields may be found from (1) and are thus given in terms of the unknown $\phi_i(x,y)$. Parseval's theorem may be applied, however, to obtain

$$P_{\text{avg}} = \frac{1}{4\pi} \operatorname{Re} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\mathcal{E}_x(\alpha,y)\mathcal{H}_y^*(\alpha,y) - \mathcal{E}_y(\alpha,y)\mathcal{H}_x^*(\alpha,y)] \, dy \, d\alpha \tag{8}$$

where the script quantities denote the transforms of the fields and are given in terms of the $\Phi_i(\alpha,y)$. At this point the general solutions for the Φ_i may be substituted and integration with respect to y can be accomplished analytically. This leaves an equation of the form

$$P_{\text{avg}} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} g(\alpha) \, d\alpha \tag{9}$$

which is evaluated numerically in each of the two regions.

Fig. 2 shows computed results for a single strip and Fig. 6 shows the characteristic impedances of the odd and even modes of coupled strips. The results again agree well with those from [6] where there is overlap. The increase of characteristic impedance with frequency, which appears to have been first discovered by Krage and Haddad, is verified by the present analysis.

There is some latitude in the definition of characteristic impedance for a structure such as microstrip. Another possible definition of characteristic impedance is given by

$$Z_{0v} = \frac{V^2(0)}{2P_{\text{avg}}} \tag{10}$$

for a single strip where $V(0)$ is given by

$$V(0) = -\int_0^D E_y(0,y) \, dy \tag{11}$$

and is the voltage between the center of the strip and the ground plane. It is interesting to compare the results

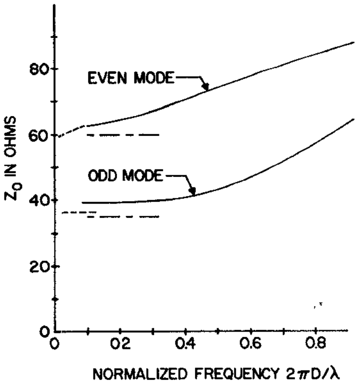

Fig. 6. Characteristic impedance versus frequency for coupled microstrips. $\epsilon_r = 10$, $W/D = 1$, $S/D = 0.4$. Present method: ——; Krage and Haddad: ——; Bryant and Weiss: – –.

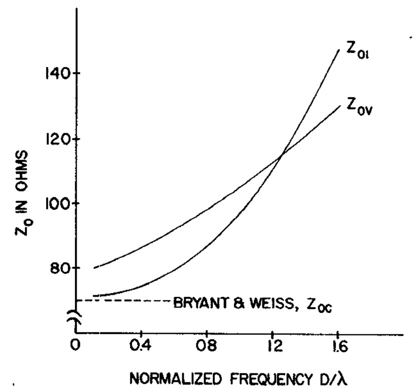

Fig. 7. Characteristic impedance versus frequency for a single microstrip using several definitions of impedance. $\epsilon_r = 9$, $W/D = 0.5$.

obtained using (11) with those obtained using (6a). Fig. 7 shows results for a single strip. Also shown is the impedance from [8]. It is evident that the definition of impedance based upon strip current converges to the quasi-static characteristic impedance Z_{0c} which is defined in terms of static capacitance. In all probability, the reason that Z_{0v} does not converge to Z_{0c} is that $V(0)$ is sensitive to the assumed distribution of surface current while the total current I_{0z} used in (6a) is not. A better approximation (recall a uniform distribution was assumed) to the surface current such as $\mathcal{J}_z(x) = [(W/2)^2 - x^2]^{-1/2}, |x| < W/2$, would probably improve the result obtained using (10). Finally, it is to be noted that the geometric mean of the two curves in Fig. 7 gives the characteristic impedance defined by $V(0)/I_{0z} = (Z_{0v}Z_{0i})^{1/2}$.

IV. CONCLUSIONS

A spectral-domain technique for analysis of single and coupled microstrips has been described. It has been shown that this method can be successfully implemented computationally to accurately calculate characteristic impedance as well as wavelength as a function of frequency. The computational efficiency is quite good and, for a given frequency, the computations (wavelength and impedance) take about 2.5 s on the IBM 360.

The numerical results presented here have been shown to be in agreement with those of other investigators at low frequencies. Considerable departure from the quasi-static results has been shown to occur with increasing frequency, however. The analysis verifies the rise of characteristic impedance with frequency as predicted by Krage and Haddad [6].

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Circular Waveguide with Sinusoidally Perturbed Walls

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Abstract—**Uniform second-order asymptotic expansions are obtained for the propagation of TM waves in a perfectly conducting circular waveguide with sinusoidally perturbed walls using the method of multiple scales. The analysis concerns the interaction of two propagating modes satisfying the resonance condition imposed by the periodicity of the waveguide walls. Two cases of resonance are treated as well as the case of decoupled modes. In the first case resonance occurs whenever the difference between the wavenumbers of the two interacting modes is nearly equal to the wall wavenumber, while in the second case the difference is nearly equal to twice the wall wavenumber. The results of the theory are then applied to the design of a mode coupler.**

I. INTRODUCTION

WAVEGUIDES having periodic structure find application in such microwave devices as the magnetron, the traveling-wave amplifier, and the linear accelerator [1]. In this paper we consider the case where the periodicity is a small perturbation of the waveguide wall which results in its use as a mode coupler.

We consider the case of propagation of TM modes in a perfectly conducting circular waveguide whose wall is sinusoidally perturbed so that the radius of the cross section of the guide at an axial location z' in a cylindrical coordinate system (ρ',ϕ,z') is given by

$$R(z') = R_0(1 + \epsilon \sin k_w'z') \tag{1}$$

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where R_0 is the average or unperturbed radius of the guide, k_w' is the wavenumber of the wall perturbation, and ϵ is a dimensionless parameter much smaller than unity and equal to the ratio of the amplitude of the periodic perturbation to the average radius R_0.

Marcuse and Derosier [2] treated the problem of a round dielectric waveguide with periodic wall corrugations and found that two guided modes are coupled if the difference between their wavenumbers is equal to the wavenumber of the wall k_w'. In fact, other resonances are possible as our analysis will show. They also confirmed the coupling experimentally and observed complete power conversion between the two modes. Marcuse used a combination of the Galerkin procedure and the method of averaging in order to obtain equations for the amplitudes of the interacting modes [3]. Chandezon *et al.* [4] treated wave propagation in a perfectly conducting guide with sinusoidally perturbed walls using the Rayleigh–Schrodinger technique to find a perturbation expansion in powers of ϵ for the case of a cylindrically symmetric TM mode $(\partial/\partial\phi = 0)$. They only considered the very special case of resonance when the wavenumber of the excited mode is equal to $\frac{1}{2}k_w'$ with a correction to first order in ϵ only.

A first-order uniformly valid expansion for the case of a parallel-plate waveguide with sinusoidal walls was obtained by Nayfeh and Asfar [5] for two interacting modes in the neighborhood of resonance that is given by the condition

$$k_w' \approx k_n' - k_s' \tag{2}$$